Stochastic Comparisons for Multivariate Shock Models

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Consider two devices subjected to shocks arriving according to two identically defined counting processes. Let N_1 and N_2 be the random numbers of shocks until failure of the two devices, respectively, and let T_1 and T_2 be their random lifetimes. Conditions such that stochastic orders between N_1 and N_2 are preserved by T_1 and T_2 have been investigated in recent literature. Here we study this problem in a multivariate setting, considering systems of non-independent components, and we extend some known results to multivariate stochastic orders. Two kinds of multivariate generalizations are considered; the case that each one of the components is subjected to its own fond of shocks and the case that all the components of the same system are subjected to a common font of shocks.

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1. INTRODUCTION

Consider two devices subjected to shocks occurring randomly in time as events of two counting processes which are independent and identically distributed. Let the integer-valued independent variables N_1 and N_2 be the random numbers of shocks that cause the damage to the two devices, and let $\mathbf{X}_1 = \{X_1^{(j)}, j \in \mathbb{N}^+\}$ and $\mathbf{X}_2 = \{X_2^{(j)}, j \in \mathbb{N}^+\}$ be the sequences of the inter-times between shocks. Then the variables

$$T_1 = \sum_{j=1}^{N_1} X_1^{(j)} \qquad \text{and} \qquad T_2 = \sum_{j=1}^{N_2} X_2^{(j)} \tag{1.1}$$

represent the random lifetimes of the two devices (here $\sum_{j=1}^{0} x^{(j)} = 0$).

Because of the large number of applied sciences where random sums of the kind in (1.1) are used, considerable attention has been devoted in recent literature to the study of properties of these random sums. In particular, different authors considered conditions such that stochastic orders

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between N_1 and N_2 are preserved by stochastic orders between T_1 and T_2. Results on this topic are discussed, for instance, in Singh and Jain [14], Pellerey [11], Kebir [8], Jean-Marie and Liu [3], and Alzaid *et al.* [1].

Here we extend this problem to the multivariate case, i.e., we consider two multicomponent systems in which the m components of each system ($m \in \mathbb{N}^+$) have non-independent tolerances to shocks. In other words, if we denote as $\mathbf{N}_1 = (N_{1,1}, ..., N_{1,m})$ and $\mathbf{N}_2 = (N_{2,1}, ..., N_{2,m})$ the vectors of the random numbers of shocks until failure of the components of the two systems, and as $\mathbf{T}_1 = (T_{1,1}, ..., T_{1,m})$ and $\mathbf{T}_2 = (T_{2,1}, ..., T_{2,m})$ the vectors of the random lifetimes of the components of the two systems, we describe herein conditions under which some multivariate stochastic comparisons between $\mathbf{N}_1$ and $\mathbf{N}_2$ are preserved into stochastic comparisons between $\mathbf{T}_1$ and $\mathbf{T}_2$.

For this purpose we define a multivariate generalization of the univariate shock model (i.e., of the random sums in (1.1)) assuming each of the m components of each system to be subjected to its own font of shocks. Then, denoting as $\mathbf{X}_{s,i} = \{X_{s,i}^{(j)}, j \in \mathbb{N}^+\}$, $s = 1, 2$ and $i \in I = \{1, ..., m\}$, the sequence of the interarrivals between the shocks that stroke the ith component of the sth system, the vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ can be represented as

$$\mathbf{T}_s = (T_{s,1}, ..., T_{s,m}) = \left(\sum_{j=1}^{N_{s,1}} X_{s,1}^{(j)}, ..., \sum_{j=1}^{N_{s,m}} X_{s,m}^{(j)} \right), \qquad s = 1, 2. \tag{1.2}$$

Interesting results on the preservation of multivariate stochastic orders for this model have been shown by Wong [15], who considered shocks occurring as events of Poisson processes. Here, essentially, we generalize his results to more general underlying counting processes. It must be pointed out that Wong [15] proved them by explicit expression of the joint survival functions of $\mathbf{T}_s$, $s = 1, 2$ while here the proofs are based on comparisons of expectations of functions of $\mathbf{T}_s$, and therefore are based on representation (1.2). As we will see, this method allows extremely simple proofs.

Different stochastic orders will be considered here. Since most of them are well-known, to abridge the paper we will not include their definitions here; we refer the reader to the book by Shaked and Shanthikumar [13, Chaps. 4 and 5], for definitions and applications.

In the following Section 2 we describe conditions for the stochastic comparisons of $\mathbf{T}_1$ and $\mathbf{T}_2$ when they are defined as in (1.2), while in Section 3 we briefly consider a different multivariate generalization of the univariate shock model, obtained assuming that all the components of each system are subjected to a common font of shocks. Note that, according to most of the reliability literature, throughout this paper we write "increasing" instead of "non-decreasing" and "decreasing" instead of "non-increasing".

Also, for any (multivariate) variable Z and an event A, we will denote by $[Z \mid A]$ any random variable whose distribution is the conditional distribution of Z given A.

2. PRESERVATION RESULTS

For all $s=1, 2$, let $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ be the vector of the infinite sequences of inter-times between shocks (thus of sequences of non-negative real-valued random variables), and let $\mathbf{N}_s$ be the vector of the numbers of shocks until failure of the components of the system s (thus of positive integer-valued random variables). Also, let the vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ be defined as in (1.2).

As in Wong [15], we will assume throughout this section independence among the vectors $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ and the vectors $\mathbf{N}_s$, for $s=1, 2$, and we will assume that the underlying counting processes of the two systems are identically distributed, i.e., that

$$(\mathbf{X}_{1,1}, ..., \mathbf{X}_{1,m}) =_{st} (\mathbf{X}_{2,1}, ..., \mathbf{X}_{2,m}).$$

The first result stated below involves the *usual stochastic order* (denoted $\leqslant_{st}$). Because of its almost trivial proof, we do not claim originality for it.

THEOREM 2.1. *For all the vectors $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ of infinite sequences of non-negative random variables, if $\mathbf{N}_1 \leqslant_{st} \mathbf{N}_2$ then $\mathbf{T}_1 \leqslant_{st} \mathbf{T}_2$.*

Proof. The proof is based on the characterization of the usual stochastic order by construction on the same probability space.

Because of assumption $(\mathbf{X}_{1,1}, ..., \mathbf{X}_{1,m}) =_{st} (\mathbf{X}_{2,1}, ..., \mathbf{X}_{2,m})$ there exist two random vectors of sequences $(\hat{\mathbf{X}}_{s,1}, ..., \hat{\mathbf{X}}_{s,m})$ defined on the same probability space and such that

$$(\hat{\mathbf{X}}_{s,1}, ..., \hat{\mathbf{X}}_{s,m}) =_{st} (\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m}), \qquad s=1, 2,$$

and

$$(\hat{\mathbf{X}}_{1,1}, ..., \hat{\mathbf{X}}_{1,m}) = (\hat{\mathbf{X}}_{2,1}, ..., \hat{\mathbf{X}}_{2,m}) \qquad \text{with probability 1.}$$

Because of assumption $\mathbf{N}_1 \leqslant_{st} \mathbf{N}_2$ there exist two random vectors $\hat{\mathbf{N}}_s$, $s=1, 2$, defined on the same probability space and such that

$$\hat{\mathbf{N}}_s =_{st} \mathbf{N}_s, \qquad s=1, 2$$

and

$$\hat{\mathbf{N}}_1 \leqslant \hat{\mathbf{N}}_2 \qquad \text{with probability 1.}$$

Because of the independence between the vectors $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ and the vectors $\mathbf{N}_s$, we can assume that the $(\hat{\mathbf{X}}_{s,1}, ..., \hat{\mathbf{X}}_{s,m})$ and the $\hat{\mathbf{N}}_s$ are all defined on the same probability space.

Let now

$$\hat{\mathbf{T}}_s = (\hat{T}_{s,1}, ..., \hat{T}_{s,m}) = \left(\sum_{j=1}^{\hat{N}_{s,1}} \hat{X}_{s,1}^{(j)}, ..., \sum_{j=1}^{\hat{N}_{s,m}} \hat{X}_{s,m}^{(j)} \right), \qquad s = 1, 2.$$

Since $\hat{\mathbf{T}}_s$ is increasing in each argument of $\hat{\mathbf{N}}_s$ (because the $\hat{X}_{s,m}^{(j)}$ are non-negative), it follows that

$$\hat{\mathbf{T}}_1 \leqslant \hat{\mathbf{T}}_2 \qquad \text{with probability 1.}$$

Obviously it is $\hat{\mathbf{T}}_s =_{st} \mathbf{T}_s$, $s = 1, 2$, and therefore $\mathbf{T}_1 \leqslant_{st} \mathbf{T}_2$. ▮

Remark 2.1. Theorem 2.1 can be generalized to the case in which the vectors $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ are not identically distributed. The proof above in fact continues to hold under the assumption

$$(\mathbf{X}_{1,1}, ..., \mathbf{X}_{1,m}) \leqslant_{st} (\mathbf{X}_{2,1}, ..., \mathbf{X}_{2,m}),$$

where the stochastic comparison $\leqslant_{st}$ between vectors of infinite sequences of random variables is defined in the usual way by comparison of expected valued of increasing functionals of the vectors, or by using the construction on the same probability space (see Kamae *et al.* [4] on this aim).

The second result involve the *convex order* (denotes as $\leqslant_{cx}$). Since it is harder to work with the convex order (if compared with the usual stochastic order) the proof above cannot be adapted to this kind of comparison, and some assumptions on the counting processes must be stated. Also, a preceding lemma is needed.

LEMMA 2.2. *Let the components of the vectors $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ be independent, and let the sequences $\mathbf{X}_{s,i} = \{X_{s,i}^{(j)}, j \in \mathbb{N}^+\}$ be of independent nonnegative random variables that are increasing in convex order (i.e., be such that $X_{s,i}^{(j)} \leqslant_{cx} X_{s,i}^{(j+1)}$ for all $s = 1, 2$, $i \in I$ and $j \in \mathbb{N}^+$). Then*

$$\left(\sum_{j=1}^{n_1-k_1} X_{s,1}^{(j)} + \sum_{j=n_1+1}^{n_1+k_1} X_{s,1}^{(j)}, ..., \sum_{j=1}^{n_m-k_m} X_{s,m}^{(j)} + \sum_{j=n_m+1}^{n_m+k_m} X_{s,m}^{(j)} \right)$$
$$\geqslant_{cx} \left(\sum_{j=1}^{n_1} X_{s,1}^{(j)}, ..., \sum_{j=1}^{n_m} X_{s,m}^{(j)} \right) \tag{2.1}$$

for all $s = 1, 2$, $n_i \in \mathbb{N}^+$ and $k_i \in \mathbb{N}$ such that $n_i - k_i \geqslant 0$ $(i \in I)$.

Proof. Observe that the random vectors in (2.1) have independent components, and therefore the stochastic inequality (2.1) holds if (and only if) all the m components are comparable in convex order (see Theorem 5.A.6 in Shaked and Shanthikumar [13]). Therefore it is enough to show that the assumptions yield

$$X_{s,i}^{(1)}+\cdots+X_{s,i}^{(n_i-k_i)}+X_{s,i}^{(n_i+1)}+\cdots+X_{s,i}^{(n_i+k_i)}\geqslant_{cx}X_{s,i}^{(1)}+\cdots+X_{s,i}^{(n_i)} \tag{2.2}$$

for all s, i, n_i and k_i as in (2.1).

Fix $i\in I$ and $s=1$ or 2. To prove (2.2) denote

$$\begin{aligned} Y_1&=X_{s,i}^{(1)}+\cdots+X_{s,i}^{(n_i-k_i)}\\ Y_2&=X_{s,i}^{(n_i-k_i+1)}+\cdots+X_{s,i}^{(n_i)}\\ Y_3&=X_{s,i}^{(n_i+1)}+\cdots+X_{s,i}^{(n_i+k_i)}. \end{aligned}$$

Thus (2.2) is equivalent to

$$Y_1+Y_3\geqslant_{cx}Y_1+Y_2. \tag{2.2a}$$

Note that, by the stochastic monotonicity in convex order of the interarrivals, it holds

$$Y_3\geqslant_{cx}Y_2. \tag{2.2b}$$

Inequality (2.2a) follows now from the independence among Y_1, Y_2, and Y_3, inequality (2.2b), and inequality (2.A.21) in Shaked and Shanthikumar [13]. ▮

We can now prove the main result for the convex order.

THEOREM 2.3. *Let the vectors* $(\mathbf{X}_{s,1},\ldots,\mathbf{X}_{s,m})$ *satisfy the assumptions of Lemma* 2.2. *Then* $\mathbf{N}_1\leqslant_{cr}\mathbf{N}_2$ *implies* $\mathbf{T}_1\leqslant_{cx}\mathbf{T}_2$.

Proof. Let ϕ be a convex real function on $\mathbb{R}^m$, and consider the function

$$\psi(\mathbf{n})=\psi(n_1,\ldots,n_m)=\mathbf{E}\left[\phi\left(\sum_{j=1}^{n_1}X_{s,1}^{(j)},\ldots,\sum_{j=1}^{n_m}X_{s,m}^{(j)}\right)\right],\qquad s=1,2$$

(note that ψ does not depend on s because $\mathbf{X}_{1,i}=_{st}\mathbf{X}_{2,i}$ for every $i\in I$). This function is convex, i.e., it is

$$\psi(\mathbf{n}+\mathbf{k})-\psi(\mathbf{n})\geqslant\psi(\mathbf{n})-\psi(\mathbf{n}-\mathbf{k})$$

for all $\mathbf{n},\mathbf{k}\in\mathbb{N}^m$ such that $\mathbf{n}-\mathbf{k}\geqslant\mathbf{0}$.

In fact, observe that for fixed **n**, **k** as above it holds

$$
\begin{aligned}
&\psi(\mathbf{n}+\mathbf{k})-\psi(\mathbf{n})\\
&\quad=\mathbf{E}\left[\phi\left(\sum_{j=1}^{n_1+k_1} X_{s,1}^{(j)},\ldots,\sum_{j=1}^{n_m+k_m} X_{s,m}^{(j)}\right)-\phi\left(\sum_{j=1}^{n_1} X_{s,1}^{(j)},\ldots,\sum_{j=1}^{n_m} X_{s,m}^{(j)}\right)\right]\\
&\quad\geqslant\mathbf{E}\left[\phi\left(\sum_{j=1}^{n_1-k_1} X_{s,1}^{(j)}+\sum_{j=n_1+1}^{n_1+k_1} X_{s,1}^{(j)},\ldots,\sum_{j=1}^{n_m-k_m} X_{s,m}^{(j)}+\sum_{j=n_m+1}^{n_m+k_m} X_{s,1}^{(j)}\right)\right.\\
&\qquad\left.-\phi\left(\sum_{j=1}^{n_1-k_1} X_{s,1}^{(j)},\ldots,\sum_{j=1}^{n_m-k_m} X_{s,m}^{(j)}\right)\right] \qquad (2.3)
\end{aligned}
$$

since the arguments of ϕ in the right hand of the inequality (2.3) are a.s. smaller than the ones in the left hand, and because of the convexity of ϕ.

Therefore

$$
\begin{aligned}
&[\psi(\mathbf{n}+\mathbf{k})-\psi(\mathbf{n})]-[\psi(\mathbf{n})-\psi(\mathbf{n}-\mathbf{k})]\\
&\quad=\mathbf{E}\left[\phi\left(\sum_{j=1}^{n_1+k_1} X_{s,1}^{(j)},\ldots\right)-\phi\left(\sum_{j=1}^{n_1} X_{s,1}^{(j)},\ldots\right)\right.\\
&\qquad\left.-\phi\left(\sum_{j=1}^{n_1} X_{s,1}^{(j)},\ldots\right)+\phi\left(\sum_{j=1}^{n_1-k_1} X_{x,1}^{(j)},\ldots\right)\right]\\
&\quad\geqslant\mathbf{E}\left[\phi\left(\sum_{j=1}^{n_1-k_1} X_{s,1}^{(j)}+\sum_{j=n_1+1}^{n_1+k_1} X_{s,1}^{(j)},\ldots\right)-\phi\left(\sum_{j=1}^{n_1-k_1} X_{s,1}^{(j)},\ldots\right)\right.\\
&\qquad\left.-\phi\left(\sum_{j=1}^{n_1} X_{s,1}^{(j)},\ldots\right)+\phi\left(\sum_{j=1}^{n_1-k_1} X_{s,1}^{(j)},\ldots\right)\right]\\
&\quad=\mathbf{E}\left[\phi\left(\sum_{j=1}^{n_1-k_1} X_{s,1}^{(j)}+\sum_{j=n_1+1}^{n_1+k_1} X_{s,1}^{(j)},\ldots\right)\right]-\mathbf{E}\left[\phi\left(\sum_{j=1}^{n_1} X_{s,1}^{(j)},\ldots\right)\right]\\
&\quad\geqslant 0,
\end{aligned}
$$

where the first inequality follows from (2.3), and the second one from Lemma 2.2.

Now the thesis observing that for every convex real function ϕ on $\mathbb{R}^m$ it is

$$
\begin{aligned}
\mathbf{E}[\phi(\mathbf{T}_1)]&=\mathbf{E}[\mathbf{E}[\phi(\mathbf{T}_1)\mid\mathbf{N}_1]]\\
&=\mathbf{E}[\psi(\mathbf{N}_1)]\\
&\leqslant\mathbf{E}[\psi(\mathbf{N}_2)]\\
&=\mathbf{E}[\mathbf{E}[\phi(\mathbf{T}_2)\mid\mathbf{N}_2]]=\mathbf{E}[\phi(\mathbf{T}_2)],
\end{aligned}
$$

where the inequality follows from the convexity of ψ and the assumption $\mathbf{N}_1 \leqslant_{cx} \mathbf{N}_2$. ▮

The preservation of other multivariate stochastic orders for the model considered in this section can be shown using arguments similar to the ones in the proof of Theorem 2.3. This is the case of the *increasing convex order* (denoted $\leqslant_{icx}$), the *componentwise convex order* (denoted $\leqslant_{ccx}$), the *increasing componentwise convex order* (denoted $\leqslant_{iccx}$), the *symmetric convex order* (denoted $\leqslant_{symcx}$), and the *symmetric increasing convex order* (denoted $\leqslant_{symicx}$). Namely, the following result holds.

THEOREM 2.4. *Let the components of the vectors* $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ *be independent, and let the sequences* $\mathbf{X}_{s,i} = \{X_{s,i}^{(j)}, j \in \mathbb{N}^+\}$ *be of independent nonnegative random variables that are increasing in icx [cx, icx, cx, icx] order (i.e., be such that* $X_{s,i}^{(j)} \leqslant_{icx} [\leqslant_{cx}, \leqslant_{icx}, \leqslant_{cx}, \leqslant_{icx}]\, X_{s,i}^{(j+1)}$ *for all* $s = 1, 2$, $i \in I$ *and* $j \in \mathbb{N}^+$*). Then*

$$\mathbf{N}_1 \leqslant_{icx} [\leqslant_{ccx}, \leqslant_{iccx}, \leqslant_{symcx}, \leqslant_{symicx}]\, \mathbf{N}_2$$

implies

$$\mathbf{T}_1 \leqslant_{icx} [\leqslant_{ccx}, \leqslant_{iccx}, \leqslant_{symcx}, \leqslant_{symicx}]\, \mathbf{T}_2.$$

Two other interesting stochastic orders, which are different multivariate extensions of the univariate usual stochastic order, are the *upper orthant order* and *lower orthant order* (denoted $\leqslant_{uo}$ and $\leqslant_{lo}$, respectively). In this case the related result is easy to prove.

THEOREM 2.5. *For all the vectors* $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ *of infinite sequences of non-negative random variables, if* $\mathbf{N}_1 \leqslant_{uo} [\leqslant_{lo}]\, \mathbf{N}_2$ *then* $\mathbf{T}_1 \leqslant_{uo} [\leqslant_{lo}]\, \mathbf{T}_2$.

Proof. We give the proof for the $\leqslant_{uo}$ case, the other being the same.

Consider m univariate increasing functions g_i, $i \in I$, fix a vector $(\mathbf{x}_1, ..., \mathbf{x}_m)$ of sequences $\mathbf{x}_i = \{x_i^{(j)} \geqslant 0, j \in \mathbb{N}^+\}$, $i \in I$, and note that

$$\prod_{i=1}^{m} g_i\left(\sum_{j=1}^{n_i} x_i^{(j)}\right) = \prod_{i=1}^{m} h_i(n_i),$$

where, obviously, the functions $h_i(n) = g_i(\sum_{j=1}^{n} x_i^{(j)})$ are increasing when $x_i^{(j)} \geqslant 0$ for all $i \in I$ and $j \in \mathbb{N}^+$. Therefore for every vector $(\mathbf{x}_1, ..., \mathbf{x}_m)$ of sequences $\mathbf{x}_i = \{x_i^{(j)} \geqslant 0, j \in \mathbb{N}^+\}$, $i \in I$, it is

$$
\begin{aligned}
\mathbf{E}\left[\prod_{i=1}^{m} g_i(T_1, i) \mid (\mathbf{X}_{1,1}, ..., \mathbf{X}_{1,m}) = (\mathbf{x}_1, ..., \mathbf{x}_m)\right]
&= \mathbf{E}\left[\prod_{i=1}^{m} g_i\left(\sum_{j=1}^{N_{1,i}} x_i^{(j)}\right)\right] \\
&= \mathbf{E}\left[\prod_{i=1}^{m} h_i(N_{1,i})\right] \\
&\leqslant \mathbf{E}\left[\prod_{i=1}^{m} h_i(N_{2,i})\right] \\
&= \mathbf{E}\left[\prod_{i=1}^{m} g_i\left(\sum_{j=1}^{N_{2,i}} x_1^{(j)}\right)\right] \\
&= \mathbf{E}\left[\prod_{i=1}^{m} g_i(T_{2,i}) \mid (\mathbf{X}_{2,1}, ..., \mathbf{X}_{2,m}) = (\mathbf{x}_1, ..., \mathbf{x}_m)\right],
\end{aligned}
$$

where the inequality follows by the assumption $\mathbf{N}_1 \leqslant_{uo} \mathbf{N}_2$ and the monotonicity of the functions h_i, $i \in I$.

It follows that for every set of increasing real function g_i on $\mathbb{R}$ it is

$$
\begin{aligned}
\mathbf{E}\left[\prod_{i=1}^{m} g_i(T_{1,i})\right] &= \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^{m} g_i(T_{1,i}) \mid (\mathbf{X}_{1,1}, ..., \mathbf{X}_{1,m})\right]\right] \\
&\leqslant \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^{m} g_i(T_{2,i}) \mid (\mathbf{X}_{2,1}, ..., \mathbf{X}_{2,m})\right]\right] \\
&= \mathbf{E}\left[\prod_{i=1}^{m} g_i(T_{2,i})\right],
\end{aligned}
$$

which is the assertion of the theorem. ▮

Like the usual stochastic order, the convex order also has different multivariate extensions. Two of them are the *upper orthant convex order* and the *lower orthant convex order*, denoted here as $\leqslant_{uo-cx}$ and $\leqslant_{lo-cv}$, respectively. The conditions for their preservation under construction of multivariate shock models are similar to the ones stated in Theorem 2.3.

THEOREM 2.6. *Let the components of the vectors* $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ *be independent, and let the sequences* $\mathbf{X}_{s,i} = \{X_{s,i}^{(j)}, j \in \mathbb{N}^+\}$ *be of independent non-negative random variables that are increasing [decreasing] in increasing convex order [decreasing concave order] (i.e., be such that* $X_{s,i}^{(j)} \leqslant_{icx} [\geqslant_{icv}] \mathbf{X}_{s,i}^{(j+1)}$ *for all* $s = 1, 2$, $i \in I$ *and* $j \in \mathbb{N}^+$*). Then* $\mathbf{N}_1 \leqslant_{uo-cx} [\leqslant_{lo-cv}] \mathbf{N}_2$ *implies* $\mathbf{T}_1 \leqslant_{uo-cx} [\leqslant_{lo-cv}] \mathbf{T}_2$.

Proof. Again, we given the proof for the $\leqslant_{uo-cx}$ case, the other being the same.

Consider m univariate increasing convex functions g_i, $i \in I$, fix a vector $(n_1, ..., n_m)$ and note that, because of the independence among the sequence $\mathbf{X}_{s,i} = \{X_{s,i}^{(j)}, j \in \mathbb{N}^+\}$, we can write

$$\mathbf{E}\left[\prod_{i=1}^{m} g_i\left(\sum_{j=1}^{n_i} X_{s,i}^{(j)}\right)\right] = \prod_{i=1}^{m} \mathbf{E}\left[g_i\left(\sum_{j=1}^{n_i} X_{s,i}^{(j)}\right)\right]$$

$$= \prod_{i=1}^{m} h_i(n_i)$$

(note that the functions $h_i(n) = \mathbf{E}[g_i(\sum_{j=1}^{n} X_{s,i}^{(j)})]$ do not depend on s since $\mathbf{X}_{1,i} =_{st} \mathbf{X}_{2,i}$).

Reasoning as in the proof of Theorem 2.3, it is not hard to verify that the functions h_1 are increasing and convex.

Therefore we have

$$\begin{aligned}
\mathbf{E}\left[\prod_{i=1}^{m} g_i(T_{1,i})\right] &= \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^{m} g_i\left(\sum_{j=1}^{N_{1,i}} X_{1,i}^{(j)}\right)\middle|\mathbf{N}_1\right]\right] \\
&= \mathbf{E}\left[\prod_{i=1}^{m} h_i(N_{1,i})\right] \\
&\leqslant \mathbf{E}\left[\prod_{i=1}^{m} h_i(N_{2,i})\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^{m} g_i\left(\sum_{j=1}^{N_{2,i}} X_{2,i}^{(j)}\right)\middle|\mathbf{N}_2\right]\right] \\
&= \mathbf{E}\left[\prod_{i=1}^{m} g_i(T_{2,i})\right],
\end{aligned}$$

where the inequality follows from the assumption $\mathbf{N}_1 \leqslant_{uo-cx} \mathbf{N}_2$ and the monotonicity and convexity of the functions h_i, $i \in I$.

The assertion follows. ∎

In the next result we consider the *Laplace transform order*, denoted $\leqslant_{Lt}$.

THEOREM 2.7. *Let the components of the vectors* $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ *be independent, and let the sequences* $\mathbf{X}_{s,i} = \{X_{s,i}^{(j)}, j \in \mathbb{N}^+\}$ *be of independent and identically distributed non-negative random variables (i.e., be such that* $X_{s,i}^{(j)} =_{st} X_{s,i}^{(j+1)}$ *for all* $s = 1, 2$, $i \in I$ *and* $j \in \mathbb{N}^+$*). Then* $\mathbf{N}_1 \leqslant_{Lt} \mathbf{N}_2$ *implies* $\mathbf{T}_1 \leqslant_{Lt} \mathbf{T}_2$.

Proof. We must show that, for every vector $\mathbf{z}=(z_1, ..., z_m)$, $\mathbf{z}\geqslant\mathbf{0}$, the assumptions yield $\mathbf{E}[\exp(-\sum_{i=1}^m z_i T_{1,i})]\geqslant\mathbf{E}[\exp(-\sum_{i=1}^m z_i, T_{2,i})]$. For it, note that

$$\begin{aligned}\mathbf{E}\left[\exp\left(-\sum_{i=1}^m z_i T_{s,i}\right)\right] &= \mathbf{E}\left[\exp\left(-\sum_{i=1}^m z_i \sum_{j=1}^{N_{s,i}} X_{s,i}^{(j)}\right)\right]\\ &= \mathbf{E}\left[\mathbf{E}\left[\exp\left(-\sum_{i=1}^m z_i \sum_{j=1}^{N_{s,i}} X_{s,i}^{(j)}\right)\middle|\mathbf{N}_s\right]\right]\\ &= \mathbf{E}[\psi_{\mathbf{v}(\mathbf{z})}(\mathbf{N}_s)],\end{aligned}$$

where

$$\begin{aligned}\psi_{\mathbf{v}(\mathbf{z})}(n_1, ..., n_m) &= \mathbf{E}\left[\exp\left(-\sum_{i=1}^m z_i \sum_{j=1}^{n_i} X_{s,i}^{(j)}\right)\right]\\ &= \mathbf{E}\left[\prod_{i=1}^m \prod_{j=1}^{n_i} \exp(-z_i X_{s,i}^{(j)})\right]\\ &= \prod_{i=1}^m \prod_{j=1}^{n_i} \mathbf{E}[\exp(-z_i X_{s,i}^{(j)})]\\ &= \prod_{i=1}^m \prod_{j=1}^{n_i} \exp(-v_i)\\ &= \prod_{i=1}^m \exp(-v_i n_i)\end{aligned}\tag{2.4}$$

for some $\mathbf{v}(\mathbf{z})=(v_1, ..., v_m)$ with $\mathbf{v}(\mathbf{z})\geqslant\mathbf{0}$. Note that the third equality follows from the independence of the variables $X_{s,i}^{(j)}$, while the fourth from the fact that they are identically distributed (for every fixed $s=1, 2$ and $i\in I$).

Now it is enough to observe that for every vector $\mathbf{z}=(z_1, ..., z_m)$ it is

$$\begin{aligned}\mathbf{E}\left[\exp\left(-\sum_{i=1}^m z_i T_{1,i}\right)\right] &= \mathbf{E}[\psi_{\mathbf{v}(\mathbf{z})}(\mathbf{N}_1)]\\ &\geqslant \mathbf{E}[\psi_{\mathbf{v}(\mathbf{z})}(\mathbf{N}_2)]\\ &= \mathbf{E}\left[\exp\left(-\sum_{i=1}^m z_i T_{2,i}\right)\right],\end{aligned}$$

where the inequality follows from the assumption $\mathbf{N}_1\leqslant_{Lt}\mathbf{N}_2$ and the identity (2.4). ∎

The next result involves the *moments order* and another multivariate stochastic order introduced in Lefévre and Picard [9]. Since this stochastic comparison is not surveyed in Shaked and Shanthikumar [13], we review its definition here. Given two $\mathbb{N}^m$-valued random vectors $\mathbf{N}_1$ and $\mathbf{N}_2$, we say that $\mathbf{N}_1$ is smaller that $\mathbf{N}_2$ in *ascending factorial moments order* than $\mathbf{N}_2$ (denoted $\mathbf{N}_1 \leqslant_{afm} \mathbf{N}_2$) if and only if

$$\mathbf{E}\left[\prod_{i=1}^{m} \binom{N_{1,i}+k_i-1}{k_i}\right] \leqslant \mathbf{E}\left[\prod_{i=1}^{m} \binom{N_{2,i}+k_i-1}{k_i}\right]$$

for all $(k_1, ..., k_m) \in \mathbb{N}^m$. It must be pointed out that the following relation holds:

$$\mathbf{N}_1 \leqslant_{moments} \mathbf{N}_2 \Rightarrow \mathbf{N}_1 \leqslant_{afm} \mathbf{N}_2.$$

More details (and applications) about this order may be found in Lefévre and Picard [9].

THEOREM 2.8. *Let the components of the vectors $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ be independent, and let the $\mathbf{X}_{s,i} = \{X_{s,i}^{(j)}, j \in \mathbb{N}^+\}$ be sequences of independent and exponentially distributed random variables with common parameter λ_i for all $s = 1, 2$, $i \in I$ and $j \in \mathbb{N}^+$ (i.e., let the underlying counting processes be homogeneous Poisson processes). Then $\mathbf{N}_1 \leqslant_{afm} \mathbf{N}_2$ implies $\mathbf{T}_1 \leqslant_{moments} \mathbf{T}_2$.*

Proof. For fixed $\mathbf{k} = (k_1, ..., k_m)$ denote

$$\begin{aligned}\psi_{\mathbf{k}}(n_1, ..., n_m) &= \mathbf{E}\left[\prod_{i=1}^{m} \left(\sum_{j=1}^{n_i} X_{s,i}^{(j)}\right)^{k_i}\right] \\ &= \prod_{i=1}^{m} \mathbf{E}\left[\left(\sum_{j=1}^{n_i} X_{s,i}^{(j)}\right)^{k_i}\right] \\ &= \prod_{i=1}^{m} \frac{k_i!}{\lambda_i^{k_i}} \binom{n_i+k_i-1}{k_i}\end{aligned}$$

(see, for instance, Marshall and Shaked [10, p. 346], for the last equality). Therefore for every fixed $\mathbf{k} = (k_1, ..., k_m)$ it is

$$\begin{aligned}\mathbf{E}\left[\prod_{i=1}^{m} (T_{1,i})^{k_i}\right] &= \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^{m} (T_{1,i})^{k_i} \mid \mathbf{N}_1\right]\right] \\ &= \mathbf{E}[\psi_{\mathbf{k}}(\mathbf{N}_1)] \\ &= \left(\prod_{i=1}^{m} \frac{k_i!}{\lambda_i^{k_i}}\right) \mathbf{E}\left[\prod_{i=1}^{m} \binom{N_{1,i}+k_i-1}{k_i}\right] \\ &\leqslant \left(\prod_{i=1}^{m} \frac{k_i!}{\lambda_i^{k_i}}\right) \mathbf{E}\left[\prod_{i=1}^{m} \binom{N_{2,i}+k_i-1}{k_i}\right] \\ &= \mathbf{E}[\psi_{\mathbf{k}}(\mathbf{N}_2)] \\ &= \mathbf{E}\left[\mathbf{E}\left[\prod_{i=1}^{m} (T_{2,i})^{k_i} \mid \mathbf{N}_2\right]\right] = \mathbf{E}\left[\prod_{i=1}^{m} (T_{1,i})^{k_i}\right],\end{aligned}$$

where the inequality follows from the assumption $\mathbf{N}_1 \leqslant_{afm} \mathbf{N}_2$. ∎

Note that as immediate consequence of Theorem 2.8 we obtain Theorem 2.7 in Wong [15], since the ascending factorial moments order is weaker than the moments order. Moreover from Theorem 2.8 also follows a preservation result for the univariate Poisson Shock model, which is stated below.

COROLLARY 2.9. *Let the sequences* $\mathbf{X}_s = \{X_s^{(j)}, j \in \mathbb{N}^+\}$ *be of independent random variables such that* $X_s^{(j)}$ *has exponential distribution with parameter* λ *for all* $s = 1, 2$ *and* $j \in \mathbb{N}^+$, *and let* N_1 *and* N_2 *be two integer valued random variables which are independent of the sequences* $\mathbf{X}_s$, $s = 1, 2$. *Then* $N_1 \leqslant_{afm} N_2$ *implies* $\sum_{j=1}^{N_1} X_1^{(j)} \leqslant_{moments} \sum_{j=1}^{N_2} X_2^{(j)}$.

The last result of this section is devoted to *likelihood ratio order*, denoted $\leqslant_{lr}$. This result is essentially a restatement of the analogous result in Wong [15], in the sense that using the same proof proposed by Wong it is possible to enlarge the assumptions as stated below. Here some notions of total positivity theory are needed; for it we refer the reader to Karlin [5] and Karlin and Rinott [7], where all of them are surveyed.

THEOREM 2.10. *Let the components of the vectors* $(\mathbf{X}_{s,1}, ..., \mathbf{X}_{s,m})$ *be independent, and let the sequences* $\mathbf{X}_{s,i} = \{X_{s,i}^{(j)}, j \in \mathbb{N}^+\}$ *be of independent non-negative random variables such that* $X_{s,i}^{(j)}$ *has logconcave density, for all* $s = 1, 2$, $i \in I$ *and* $j \in \mathbb{N}^+$. *Then* $\mathbf{N}_1 \leqslant_{lr} \mathbf{N}_2$ *implies* $\mathbf{T}_1 \leqslant_{lr} \mathbf{T}_2$.

Proof. Let p_s denote the distribution of the vector $\mathbf{N}_s$, i.e., let $\mathbf{P}[N_{s,1} = k_1, ..., N_{s,m} = k_m] = p_s(k_1, ..., k_m)$ for all $(k_1, ..., k_m) \in \mathbb{N}^m$, and let $f_{i,j}$ be the density of $X_{s,i}^{(j)}$. Also, denote with $f_i^{(k)}$ the density of $\sum_{j=1}^{k} X_{s,i}^{(j)}$, i.e. let $f_i^{(k)} = f_{i,1} * f_{i,2} \cdots * f_{i,k}$ where $*$ denotes the convolution. Then for every $s = 1, 2$, the vector $\mathbf{T}_s$ has density

$$f_s(t_1, ..., t_m) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \psi(k_1, ..., k_m, t_1, ..., t_m)\, p_s(k_1, ..., k_m),$$

where

$$\psi(k_1, ..., k_m, t_1, ..., t_m) = \prod_{i=1}^{m} f_i^{(k_i)}(t_i)$$

(here $t_i \geqslant 0$ for every $i \in I$).

By Theorem 1 in Karlin and Proschan [6] the functions $f_i^{(k)}(t)$ are TP_2 in $(k, t) \in \mathbb{N} \times \mathbb{R}^+$. It follows (as one can easily verify) that $\psi(k_1, ..., k_m, t_1, ..., t_m)$ is TP_2 in every pair of variables when the others are held fixed, and therefore that ψ is an MTP_2 function (see Karlin and Rinott [7, p. 469]).

The assertion of the result now follows from the closure property of the multivariate likelihood order with respect to the composition (see Theorem 2.4 in Karlin and Rinott [7]). ▮

3. COMPONENTS SUBJECTED TO A COMMON FONT OF SHOCKS

In this section we briefly consider a different multivariate extension of the univariate shock model, obtained assuming that the shocks simultaneously occur to all the components of each system. That is, we can think that all the components of each system are subjected to a common font of shocks. In such a case, then, the vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ can be represented as

$$\begin{aligned}\mathbf{T}_s &= (T_{s,1}, ..., T_{s,m})\\ &= \left(\sum_{j=1}^{N_{s,1}} X_s^{(j)}, ..., \sum_{j=1}^{N_{s,m}} X_s^{(j)}\right), \qquad s=1,2, \end{aligned} \tag{3.1}$$

where the sequences $\mathbf{X}_s = \{X_s^{(j)}, j \in \mathbb{N}^+\}$, for $s=1, 2$, are as in (1.1).

Note that in the literature few papers deal with this generalization of univariate shock models, perhaps because of their complex structure. To our knowledge the only references to it are Savits and Shaked [12], Marshall and Shaked [10], and Griffith [2], where multivariate aging notions of the vectors $\mathbf{T}_s$ are studied.

The preservation properties of the usual stochastic order, the upper orthant order and the lower orthant order for this model are contained, as a special case, in Theorem 2.1 and Theorem 2.5, and are stated below.

THEOREM 3.1. *For all the sequences* $\mathbf{X}_s = \{X_s^{(j)}, j \in \mathbb{N}^+\}$ *of non-negative random variables, if* $\mathbf{N}_1 \leqslant_{st} \mathbf{N}_2$ *then* $\mathbf{T}_1 \leqslant_{st} \mathbf{T}_2$.

THEOREM 3.2. *For all the sequences* $\mathbf{X}_s = \{X_s^{(j)}, j \in \mathbb{N}^+\}$ *of non-negative random variables, if* $\mathbf{N}_1 \leqslant_{uo} [\leqslant_{lo}]\, \mathbf{N}_2$ *then* $\mathbf{T}_1 \leqslant_{uo} [\leqslant_{lo}]\, \mathbf{T}_2$.

Remark 3.1. As for Theorem 2.1 the assumption $\mathbf{X}_1 =_{st} \mathbf{X}_2$ of Theorem 3.1 can be easily generalized to $\mathbf{X}_1 \leqslant_{st} \mathbf{X}_2$.

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